5[F].—H. GUPTA, M. S. CHEEMA, A. MEHTA & O. P. GUPTA. Edited by J. C. P. MILLER, *Representation of Primes by Quadratic Forms*, Royal Society Mathematical Tables No. 5, Cambridge University Press, New York, 1960, xxiv + 135 p., 29 cm. Price $8.50.

The eight tables in this work give solutions a, b and k, n to two related Diophantine equations

$$\kappa p = a^2 + Db^2 \quad \text{and} \quad \kappa kp = n^2 + D,$$

with $\kappa = 1$ and 2, $D = 5, 6, 10$, and 13, and for all primes $p < 100{,}000$ for which solutions exist. In each table there are approximately 2400 primes, that is, about one-fourth of all primes $<100{,}000$.

The subtitle indicates that this volume is Part I of a larger work and implies that other values of D will be forthcoming. The significance of these particular values, 5, 6, 10, and 13, is that in all quadratic number fields, $R(\sqrt{-D})$, these are the smallest positive D's for which unique factorization of the algebraic integers is lacking. In all four cases the class number is 2 (there are two classes of ideals) and this is associated with the two values of κ. For about one-half of the p's for which $-D$ is a quadratic residue a solution exists for $\kappa = 1$, and for the remaining one-half, for $\kappa = 2$. It may be noted that the $\kappa = 1$ table is always somewhat shorter than its companion $\kappa = 2$ table. This is as expected, since there are *generally* more primes of the form $4Dm + N$ than of the form $4Dm + R$ if R is a quadratic residue of $4D$ and N is not (see *MTAC*, v. 13, 1959, p. 272–284).

There is an interesting, twelve-page introduction to the background ideal and class number theory. It is boldly stated there, p. xii, that unique factorization exists for square-free $D > 0$ only if $D = 1, 2, 3, 7, 11, 19, 43, 67$, and 163. However, this has never been fully proven. This background theory culminates in several theorems due to H. P. F. Swinnerton-Dyer.

The tables were done by hand, "with the help of two long strips of paper—A and B," and were carefully checked in a variety of ways. Why the solutions to $kp = n^2 + 6$ for $p = 7$ and $p = 31$ are listed, on p. 36, as $k, n = 6, 6$ and $22, 26$ respectively rather than the obvious 1, 1 and 1, 5 is not clear to the reviewer. But this slip seems to be exceptional.

The tables may be used as lists of prime ideals in the four fields, $R(\sqrt{-D})$. The dust jacket suggests a second application, that they "may contribute to the understanding of such unsolved questions as 'Is the number of primes of form $n^2 + 5$, or $n^2 + 6$, etc., infinite or finite?' " In this respect, however, it may be remarked that the data here are rather meager since the primes are $<10^5$, while in 10 minutes an IBM 704 can count such primes up to $3.24 \cdot 10^{10}$ (see *MTAC*, v. 13, 1959, p. 78–86). In fact the 16th prime of the form $n^2 + 5$ is 86441 and is the largest listed in this volume, while the 4368th prime of that form is 32,371, 926, 089. Nonetheless, the data are sufficient to indicate at least rough agreement with the Hardy-Littlewood conjecture. Since

$$L_5(1) = 1.405 > L_6(1) = 1.283 > L_{10}(1) = 0.993 > L_{13}(1) = 0.871$$

we should expect the relative density of primes to increase as we progress from $n^2 + 5$ to $n^2 + 6$ to $n^2 + 10$ to $n^2 + 13$ (*Math. Comp.*, v. 14, 1960, p. 324–326). This is indeed the case.

An apparent anomaly concerned $n^2 + 13$, where there were numerous primes from $n = 0$ to 68 and from $n = 264$ to 298, but none in between. This striking maldistribution was most alarming, and threatened dire consequences to the Hardy-Littlewood conjecture, until the real explanation was found—pages 105 to 120 were missing in the reviewer's copy. Aside from this gross lapse, the volume has the usual elegance of the Royal Society Mathematical Tables.

D. S.

6[F, L].—C. B. HASELGROVE in collaboration with J. C. P. MILLER, *Tables of the Riemann Zeta Function*, Royal Society Mathematical Tables No. 6, Cambridge University Press, New York, 1960, xxiii + 80 p., 29 cm. Price $9.50.

These important and fascinating tables are concerned primarily with $\zeta(\frac{1}{2} + it)$, the zeta function for real part $\frac{1}{2}$, and with its zeros. This complex function is expressed both in cartesian and polar forms:

$$\begin{aligned}\zeta\,(\tfrac{1}{2} + it) &= \mathcal{R}\zeta\,(\tfrac{1}{2} + it) + i\mathcal{I}\zeta\,(\tfrac{1}{2} + it)\\ &= Z(t)e^{-i\pi\theta(t)}.\end{aligned}$$

In the latter, $\theta(t)$ is continuous, with $\theta(0) = 0$, and the signed modulus $Z(t)$, given by

$$Z(t) = \pi^{-\frac{1}{2}it}\frac{\Gamma(\frac{1}{4} + \frac{1}{2}it)}{|\Gamma(\frac{1}{4} + \frac{1}{2}it)|}\,\zeta(\tfrac{1}{2} + it),$$

changes sign at every zero. The nth zero, γ_n, is the nth solution of $Z(\gamma) = 0$ and the nth Gram point, g_n, is the solution of $\theta(g_n) = n$.

Table I gives $\mathcal{R}\zeta(\frac{1}{2} + it)$, $\mathcal{I}\zeta(\frac{1}{2} + it)$, $Z(t)$, and $\theta(t)$ to 6D for $t = 0(0.1)100$. $\mathcal{R}\zeta(1 + it)$ and $\mathcal{I}\zeta(1 + it)$ are also listed.

Table II gives $Z(t)$ to 6D for $t = 100(0.1)1000$.

Table III has two parts. Part 1, for $n = 1(1)650$, gives γ_n, g_{n-1}, and $\phi_n = (1/\pi)ph\zeta'(\frac{1}{2} + i\gamma_n)$ to 6D and $|\zeta'(\frac{1}{2} + i\gamma_n)|$ to 5D. Part 2, for $n = 651(1)1600$, gives γ_n to 6D and $|\zeta'(\frac{1}{2} + i\gamma_n)|$ to 5D.

Table IV gives $Z(t)$ to 6D for four other ranges of t,

$$t = 7000(0.1)7025, \quad 17120(0.1)17145, \quad 100000(0.1)100025, \quad 250000(0.1)250025.$$

Also for these four ranges are given the zeros to 6D and derivatives to 5D. There are 28, 32, 38, and 42 zeros in the four ranges respectively.

Table V gives $(1/\pi)ph\Gamma(\frac{1}{2} + it)$ to 6D for $t = 0(0.1)50(1)600(2)1000$.

The inclusion of g_n and ϕ_n in part 1 of Table III allows the reader to study Gram's "Law" which states that the zeros and Gram points are interlaced:

$$\gamma_{n-1} < g_{n-2} < \gamma_n < g_{n-1} < \gamma_{n+1}\,.$$

A violation occurs if $|\phi_n| > \frac{1}{2}$. The first violation is for $n = 127$. The first double violation is for $n = 379$ and 380—i.e., there are three Gram points between γ_{379} and γ_{380}. In all, there are 22 violations in these 650 zeros. Gram's "Law" may also be expressed by saying that the complex $\zeta(\frac{1}{2} + it)$ approaches its zero via the 4th or 3rd quadrant. Thus the following statistics for these 650 zeros are of interest: 4th quadrant, 320 cases; 3rd quadrant, 308; 2nd quadrant, 13, and 1st quadrant, 9.